5. ALTERNATIVES TO INITIALISE THE PARAMETERIZED EXPECTATIONS ALGORITHM

JEL Classification: C63, E17.

Keywords: Nonlinear models, Numerical solution methods, Parameterized Expectations algorithm, Optimal growth.

Remarks

1. A version of this chapter has been published within Economics Letters:

   Pérez, J.J. and A.J. Sánchez (2009) Alternatives to initialize the Parameterized Expectations Algorithm. *Economics Letters*, 102, pp. 116–118. (Direct link)

2. An earlier version of this chapter was presented at the XI Spring Meeting of Young Economists.

5.1 Motivation

The Parameterized Expectations Algorithm (PEA) is a widely applied method for solving non-linear stochastic dynamic models with rational expectations (den Haan and Marcet 1990, Marcet and Marshall 1994, Maliar and Maliar 2003 and 2005). The PEA scheme involves the approximation of the conditional expectation functions in the Euler equations with certain parametric functions, and the use of a numerical optimization method to find the parameter values.

A common problem with the PEA is that the algorithm, per se, does not guarantee convergence when initialised from arbitrary initial conditions. Therefore researchers have to either (1) find good initial conditions, or (2) fix the algorithm in some way to avoid non-convergent paths. Within group (1) the usual approach is based on the principles of homotopy. The basic idea is to slowly move from a simple case, where the solution is known or easy to compute, to the desired case where the solution is typically unknown. In this way, only local stability of the algorithm that solves for the fixed point is needed. For an introduction to homotopy theory see Garcia and Zangwill (1981); Eaves and Schmedders (1999) provide an introduction to using homotopies in economics. Nevertheless, depending on the complexity of the model at hand, this type of initialization methods may be of difficult and/or cumbersome application. Within group (2) would lie the proposal to initialise the PEA on the basis of log-linear approximations as in Pérez (2004) or the proposal of Maliar and Maliar (2003) (MM henceforth), based on restricting the simulated series within certain bounds. MM's intuitive idea was to rule out the possibility of (ex)implosive behavior by restricting the simulated series within certain bounds. As the solution is refined along the iterations, the bounds are gradually removed. The authors claim that: "The modified PEA can systematically converge to the stationary solution starting from the non stochastic steady state" (p. 88). In particular, Maliar and Maliar solve the basic neoclassical growth model and provide two simulations to illustrate that the modified PEA can find the stochastic solution starting from the non stochastic steady state. Nevertheless, the authors do not provide any systematic or theoretical assessment of the convergence properties of their algorithm.

Upon the basis of a Monte Carlo experiment, we check in a systematic way the properties of the different alternative methods. We have chosen two models to frame the discussion: the simple neoclassical growth model, and the Cooley and Hansen (1989) model, that adds to the previous model a non convexity, indivisible labor, and introduces money via a cash-in-advance constraint in consumption.

5.2 PEA and the moving bounds

Consider an economy which is described by a vector of n endogenous variables, z_t, and a vector of s exogenously given shocks, u_t. Let the process $\{z_t, u_t\}$ be represented by a system

$$g(E_t[\phi(z_{t+1}, z_t)], z_t, z_{t-1}, u_t) = 0, \quad \text{for all} \quad t \tag{5.1}$$

where $g : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^q$ and $\phi : \mathbb{R}^{2n} \to \mathbb{R}^m$; u_t follows a first-order Markov process. It is assumed that z_t is uniquely determined by (6.1) if the rest of arguments are given. The functions $g(\cdot)$ and $\phi(\cdot)$ are known functions once the structural parameters of the economy are fixed. Alternatively, let the solution be expressed as a law of motion h such that the vector z_t generated by $z_t = h(z_{t-1}, u_t)$ fulfills (6.1). Obtaining a solution to (6.1) using PEA consists of finding a parametric function $\psi(\beta; z_{t-1}, u_t)$, such that for a positive integer ν, $\beta \in \mathbb{D}^\nu$, where $\mathbb{D}^\nu \subset \{\beta \in \mathbb{R}^\infty : i\text{th element of } \beta \text{ is zero if } i > \nu\}$, the process $\{z_t(\beta)\}$ satisfies for all t the set of equations

$$g(\psi(\beta; z_{t-1}, u_t), z_t(\beta), z_{t-1}(\beta), u_t) = 0 \tag{5.2}$$

and the order of ν is such that when solving $G(\beta) = \arg\min_{\beta \in \mathbb{D}^\nu} E_t[\phi(z_{t+1}(\beta), z_t(\beta)) - \psi(\beta; z_{t-1}(\beta), u_t)]^2$, then $\beta = G(\beta)$. This problem is solved with a Gauss-Newton algorithm that uses the following updating rule: $\beta^i = \beta^{i-1} + \lambda G(\beta^{i-1})$ at each iteration i. Given these conditions, the stochastic process $\{z_t(\beta)\}$ is the PEA approximated solution. Under certain regularity conditions over the functions defining the equilibrium (6.1), the function $g(\cdot)$ is invertible in its second argument, and equation (6.2) can be written as $z_t(\beta) = h_\beta(z_{t-1}(\beta), u_t)$ for stationary and ergodic processes. Marcet and Marshall (1994) show that under those regularity conditions, fulfilled by standard business cycle models, it is always possible to find an approximated function $h_\beta(\cdot)$ arbitrarily close to the true law of motion of the system $h(\cdot)$. Under the true law of motion $h(z_{t-1}, u_t)$, the true process $\{z_t, u_t\}_{t=-\infty}^{\infty}$ verifying (6.1) is stationary. For the approximation to be acceptable, it is necessary that, given initial conditions $\{z_0, u_0\}$ and an initial vector β, the resulting process $\{z_t(\beta)\}_{t=1}^T$ verifying (6.2) has to be stationary.

In order to achieve stationarity in $\{z_t(\beta)\}_{t=1}^T$ starting for an arbitrary initial value for β, MM bounds artificially the solution to (6.2) to induce the stationarity of possibly (ex)implosive simulated series by not allowing such series to go beyond a fixed range $\underline{z} < z_t(\beta) < \overline{z}$. The range becomes irrelevant as the number of iterations increases. The PEA algorithm as modified by Maliar and Maliar (2003) can be written as follows:

- Step 1. Fix upper and lower bounds, $\underline{z}$ and $\overline{z}$ for the process $\{z_t(\beta)\}$. For an initial iteration $i = 0$ fix $\beta = \beta(0)$. Fix initial conditions u_0 and z_0; draw and fix a random series $\{u_t\}_{t=1}^T$ from a given definition. Replace the conditional expectation in (6.1) with a function $\psi(\beta; z_{t-1}, u_t)$ and compute $z_t(\beta)$ from (6.3).

- Step 2. For a given $\beta \in \mathbb{D}^\nu$ recursively calculate $\{z_t(\beta)\}_{t=1}^T$ according to

$$\begin{aligned} z_t(\beta) &= \underline{z}, && \text{if } z_t(\beta) < \underline{z} \\ z_t(\beta) &= \overline{z}, && \text{if } z_t(\beta) > \overline{z} \\ z_t(\beta) &= h_\beta(z_{t-1}(\beta), u_t), && \text{if } \underline{z} \leq z_t(\beta) \leq \overline{z} \end{aligned}$$

- Step 3. Find a $G(\beta)$ that satisfies $G(\beta) = \arg\min_{\xi \in \mathbb{D}^\nu} E_t[\phi(z_{t+1}(\beta), z_t(\beta)) - \psi(\xi; z_{t-1}(\beta), u_t)]^2$. In order to perform this step, one can run a nonlinear least squares regression with the sample $\{z_t(\beta), u_t\}$, taking $\phi(z_{t+1}(\beta), z_t(\beta))$ as a dependent variable, $\psi(\cdot)$ as an explanatory function, and ξ as a parameter vector to be estimated.

- Step 4. Compute the vector β^{i+1} for the next iteration,

$$\beta^{i+1} = \beta^i + \lambda G(\beta^i), \quad \lambda \in (0, 1)$$

- Step 5. Compute $\underline{z}(i+1)$ and $\overline{z}(i+1)$ for the next iteration,

$$\begin{aligned} \underline{z}(i+1) &= \underline{z}(i) - \underline{\Delta}(i) \\ \overline{z}(i+1) &= \overline{z}(i) + \overline{\Delta}(i) \end{aligned}$$

  where $\underline{\Delta}(i)$ and $\overline{\Delta}(i)$ are the corresponding steps.

Iterate on Steps $2-5$ until $\|\beta^{i+1} - \beta^i\|$ is below a certain tolerance value, and $\underline{z} < z_t(\beta^{i+1}) < \overline{z}$, $\forall t$.

5.3 The models

We have selected two models to frame the discussion: the one-sector stochastic growth model (SN model henceforth) and the Cooley and Hansen (1989) model (CH model henceforth). Consider firstly the SN model,

$$\max_{\{c_t, k_t\}_{t=0}^\infty} E_0 \sum_{t=0}^{\infty} \delta^t \frac{c_t^{1-\gamma} - 1}{1-\gamma}, \qquad s.t. \quad c_t + k_t = (1-d)k_{t-1} + \theta_t k_{t-1}^\alpha,$$

where $\log\theta_t = \rho\log\theta_{t-1} + \epsilon_t$ with $\epsilon_t \backsim N(0,\sigma^2)$, the initial condition (k_{-1},θ_0) is given. c_t is consumption at time t, k_{t-1} the beginning of period t capital stock, $0<\delta<1$ is the subjective discount factor, $0<\alpha<1$ the capital share in production, $0<d<1$ the depreciation rate, and $0<\rho<1$. But for the case with logarithmic utility, $\gamma=1$, and full depreciation of capital, $d=1$, a closed-form solution to this model is not known. Following den Haan and Marcet (1990), we approximate the conditional expectation by

$$E_t[c_{t+1}^{-\gamma}(1-d+\alpha\theta_{t+1}k_{t-1}^{\alpha})] \approxeq \exp(\beta_0+\beta_1\log\theta_t+\beta_2\log k_{t-1})$$

where $\beta=(\beta_0,\beta_1,\beta_2)$ is a vector of coefficients to be found. To simulate the model, parameter values are fixed as: $\alpha=0.33$, $\delta=0.95$, $\gamma=1$, $d=0.02$, $\rho=0.95$, $\sigma=0.01$, $k_{-1}=k_{ss}$ (the subscript ss refers to the steady-state values) and $\theta_0=1$.

The CH model is a bit more complex in that it includes a non-convexity, indivisible labor. Money is introduced via a cash-in-advance constraint in consumption. The competitive equilibrium is non-Pareto-optimal, and the second welfare theorem does not apply. The representative firm solves a standard profit maximization problem, while households seek to maximize their time preferences subject to their holdings of money balances and a set of standard budget constraints. There are two sources of uncertainty in this economy: the autoregressive shock to technology, θ_t, and an autoregressive logged money growth rate, $\log(g_t)$. den Haan and Marcet(1994)'s preferred specification for the approximating function $\psi(\cdot)$ to the expectation term is a third-order polynomial such that,

$$\begin{aligned} E_t[\mu_{t+1}(\alpha\theta_{t+1}k_t^{\alpha-1}N_{t+1}^{1-\alpha}+1-d)] \approxeq \quad & \exp\big(\ \beta_0+\beta_1\log k_{t-1}+\beta_2\log\theta_t+\beta_3\log g_t \\ & +\beta_4(\log k_{t-1})^2+\beta_5\log k_{t-1}\log\theta_t+\beta_6(\log\theta_t)^2 \\ & +\beta_7(\log\theta_t)^3\ \big) \end{aligned} \tag{5.3}$$

where μ_t is the Lagrange multiplier attached to the household's budget constraint, and N_t denote hours worked. Following den Haan and Marcet, we will adopt as baseline parameterization: $\delta=0.99$, $\alpha=0.36$, $A_N=2.86$, $\rho=0.95$, $\rho_g=0.48$, $\sigma=0.00721$, $\sigma_{\epsilon_g}=0.009$, $g_{ss}=1.15$, and $d=0.025$.

5.4 Initialization methods

MM propose a new approach based on restricting the simulated series within certain bounds. The respective bounds are defined as follows:

$$\begin{aligned} \underline{k}(i) &= k_{ss}\exp(-ai), \\ \overline{k}(i) &= k_{ss}(2-\exp(-ai)) \end{aligned} \tag{5.4}$$

where $a > 0$ determines the path at which the bounds are moved. Under this choice, from the first iteration ($i = 0$), in which both bounds coincide with the steady-state solution, the lower and upper bounds gradually move, approaching 0 and $2k_{ss}$, respectively. MM suggest to set a so that $\underline{k} = 0.5z_{ss}$ and $\overline{k} = 1.5z_{ss}$ is reached after a given number of iterations I. Substituting this rule in (5.4) gives $a = \log(2)/I$. [1]

In addition to the *standard* implementation suggested by MM we tried two variants aiming at improving its performance. First, we consider an extension (let us call it *adaptable* MM) along the lines of the concept of statistical depth functions (see Liu 1990 or Zuo and Serfling 2000): the speed of the moving bounds is made contingent on the number of censored points in the previous iteration. The reformulated bounds are defined as follows:

$$\begin{aligned} \underline{k}(i) &= k_{ss}\left(\exp(-ai) + 0.5\tfrac{HIT_l}{T}\right), \\ \overline{k}(i) &= k_{ss}\left(2-\exp(-ai) + 0.5\tfrac{HIT_u}{T}\right) \end{aligned} \tag{5.5}$$

where HIT_j stands for the number of censored points in the previous iteration (lower bound $j = l$, upper bound $j = u$). Second, we consider an extension in the case of the CH model whereby not only the endogenous state variable k is censored in each iteration, but also N.

Pérez (2004) proposes an approach based on a log-linearization of the necessary equations characterizing the equilibrium of the system. Let us denote a log-linear approximation to (6.1) as,

$$\hat{g}(E_t[\phi(\hat{z}_{t+1}, \hat{z}_t)], \hat{z}_t, \hat{z}_{t-1}, u_t) = 0, \quad \text{for all} \quad t \tag{5.6}$$

where $\hat{z}$, denotes the log-linearized counterpart of z, and $\hat{g}(\cdot)$ is a log-linear function approximating $g(\cdot)$. The stable solution to (5.6) can be obtained in a standard way by solving for the desired recursive equilibrium law of motion $\hat{z}_t(\beta) = \hat{h}(\hat{z}_{t-1}(\beta), u_t)$.

Solving for the stable manifold of the system forces the transversality conditions to hold. This, in turn, is a necessary and sufficient condition for the stationarity of the solution. Basing the obtention of the initial conditions for the non-linear model in a log-linear version about the

deterministic steady state of the very model, implicitly makes use of the ideas of homotopy. The log-linear approximation is a local counterpart of the model about the steady state, and at least locally should be close to the nonlinear model. As described in Pérez (2004), one could exploit the parallelism between the set of PEA-approximated first order conditions in (6.2) and the log-linear first order conditions in (5.6), or between the PEA-law of motion in (6.3) and the log-linear law of motion. The latter alternative is close to that suggested by Christiano and Fisher (2000).

We follow the first approach, for it is more generally applicable and presents better convergence properties (see Pérez, 2004). The method consists of evaluating (6.2) for $\hat{z}_t$, and then estimating the approximated $g_\beta(\cdot)$ and implied $h_\beta(\cdot)$ by means of nonlinear regressions. Given regressions are run between stationary variables, the resulting estimated β parameters have a stationary distribution. On other grounds, if T is long enough, the potential multicollinearity problems that might arise are kept to a minimum. Consider the Cooley and Hansen model. From a log-linear approximation one can define the variable $\hat{\psi}_t \equiv \hat{\mu}_{t+1}(\alpha\theta_{t+1}\hat{k}_t^{\alpha-1}\hat{N}_{t+1}^{1-\alpha} + 1 - d)$, and then, upon the basis of the specification of the functional form of the PEA approximating function in (6.4) run the nonlinear regression

$$\begin{aligned}\hat{\psi}_t = \quad & \exp(\ \varpi_0 + \rho_1 \log \hat{k}_{t-1} + \varpi_2 \log \theta_t + \varpi_3 \log g_t \\ & + \varpi_4 (\log \hat{k}_{t-1})^2 + \varpi_5 \log \hat{k}_{t-1} \log \theta_t + \varpi_6 (\log \theta_t)^2 + \varpi_7 (\log \theta_t)^3 + \varepsilon_t\ )\end{aligned}$$

The estimation of the ϖ parameters from the preceding regression would give a good starting point for the β coefficients needed to initialise PEA, as standard results in regression analysis guarantee a stationary distribution for the estimated vector of coefficients ϖ.

Another option is to follow the standard homotopy approach. We get the PEA solution for the SN model given by the analytical solution for the case with $d = 0.00$ as the initial point for our baseline. For the CH model we take the PEA solution to the SN model as the starting point.

We solve the two models described above with PEA, using the different alternative initialization methods, for 250 independent draws of the exogenous processes of size $T = 5000$ in each case (keeping the same shock for all alternatives in each simulation). The convergence criterion used is that the L^2 distance between vectors β obtained in two subsequent iterations is less than 10^{-5}.

5.5 Results and discussion

The computational results of the experiment are shown in Table 5.1. MATLAB codes for solving the two selected models with the selected initialization alternatives are available from the authors upon request.

In terms of total computational time, the PEA solution based on the baseline MM's approach took between two and two and a half times more in the SN model than the alternatives, and almost four and a half times in the case of the CH model. Regarding the stability of the solutions, with both models the maximum and minimum time per simulation was around two to five times that of the typical simulation with the alternatives. The same picture emerges when the standard deviation of the time to convergence of the 250 simulations is considered.[2]

For almost all shocks (except 1.2% of them) the baseline MM algorithm was able to make PEA converge to the rational expectations equilibrium in the case of the one-sector neoclassical model (with $\lambda = 1$). On the contrary, in the case of the CH model only 4.4% of the simulations converged under the standard implementation of MM (with $\lambda = 0.3$). Nevertheless, all the implemented modifications of the standard MM implementation led to an increase in the number of convergent paths. Keeping $\lambda = 0.3$, the number of converged simulations increased to 16% in the *adaptable* MM case, and to 19.2% in the *adaptable* MM case with k and N censored. In this latter case, censoring two variables instead of one, increased the number of converged paths by an additional 3.2%. Decreasing λ from 0.3 to 0.01 increased the number of converged paths from 16% to 21.2% in the *adaptable* MM case with only k censored. The alternative initialization methods made the PEA converge in all cases (estimated log-linear approach) and in all cases except some 4% (standard homotopy approach).

Obviously, we have only experimented in the context of two models, so it may well be that MM outperforms Pérez in some other class of models, however these examples show that fine-tuning the MM approach improves its convergence properties, and that further fine-tuning beyond the modifications presented in this paper might eventually provide more reasonable scores than the ones shown in Table 5.1. Nevertheless, having to devote an important amount of time fine-tuning MM's approach could eventually make this latter approach lose part of its major appeal: i.e. that even starting from a not very accurate initial guess might guarantee convergence.

5.6 References

Christiano, L. and Fisher, J.M. (2000) Algorithms for Solving Dynamic Models with Occasionally Binding Constraints. *Journal of Economic Dynamics and Control*, 24, pp. 1179-1232.

Cooley, T.F. and Hansen, G. (1989) The Inflation Tax in a Real Business Cycle Model. *American Economic Review*, 79, pp. 733-748.

den Haan, W. and Marcet, A. (1990) Solving the Stochastic Growth Model by Parameterized Expectations. *Journal of Business and Economic Statistics*, 8, pp. 31-34.

den Haan, W. and Marcet, A. (1994) Accuracy in Simulations. *Review of Economic Studies*, 61, pp. 3-17.

Fackler, P. (2005) A MATLAB solver for Nonlinear Rational Expectations Models. *Computational Economics*, 26, pp. 173-181.

Garcia, C.B. and Zangwill, W.I. (1981) *Pathways to solutions, fixed points, and equilibria*. Prentice-Hall.

Eaves, B.C. and Schmedders, K. (1999) General equilibrium models and homotopy methods. *Journal of Economic Dynamics and Control*, 23, pp. 1249-1279.

Liu, R.Y. (1990) On a Notion of Data Depth Based on Random Simplices. *The Annals of Statistics*, 18, pp. 405-414.

Maliar, S. and Maliar, L. (2003) Parameterized expectations algorithm and the moving bounds. *Journal of Business and Economic Statistics*, 1, pp. 88-92.

Maliar, S. and Maliar, L. (2005) Parameterized Expectations Algorithm: How to Solve for Labor Easily. *Computational Economics*, 25 (3), pp. 269-274.

Marcet, A. and Marshall, D.A. (1994) Solving Nonlinear Rational Expectations Models by Parameterized Expectations: Convergence to Stationary Solutions. Institute for Empirical Macroeconomics Discussion Paper 91, May.

Pérez, J.J. (2004) A Log-linear Homotopy Approach to Initialize the Parameterized Expectations Algorithm. *Computational Economics*, 24, pp. 59-75.

Zuo, Y. and Serfling, R. (2000) General Notions of Statistical Depth Function. *The Annals of Statistics*, 28, pp. 461-482.

Notes

[1]For the SN model with $\lambda = 1.0$ this rule would imply setting $a = 0.007$ as convergence is achieved after approximately 100 iterations ($I = 100$). For the CH model a fine tuning of λ was needed in order to find convergent paths in the case of MM. For a lower value such as $\lambda = 0.3$ the parameter a is set to 0.0035 (it corresponds to having $\underline{k} = 0.5z_{ss}$ and $\overline{k} = 1.5z_{ss}$ after $I = 200$). For even lower values like $\lambda = 0.01$ the resulting value for a had to be reduced to $a = \log(2)/2000$.

[2]All converged simulations (irrespective of the initialization method) reached the same numerical values for the set of parameters of the PEA polynomials (for each realization of the exogenous shocks). Regarding the *rationality* of the converged solutions it is worth noticing that den Haan and Marcet (1994) showed that: (i) in the case of the SN model a first order polynomial does not pass their rationality test (in this case we kept this latter parameterization for the sake of comparison with MM); (ii) in the case of the CH model the third order polynomial passes their rationality test.

5.8 Tables

Tab. 5.1: Comparison of initialization alternatives. Computational results

One-sector stochastic growth model

	Maliar and Maliar: *standard* implementation k censored $\lambda = 1$, $a = 0.007$	Maliar and Maliar: *adaptable* moving bounds k censored $\lambda = 1$, $a = 0.007$	Estimated Log-linear approach $\lambda = 1$	Standard Homotopy approach $\lambda = 1$
Average computational time	2.649	2.552	1.000	1.378
Maximum time simulation	2.835	2.716	1.000	1.407
Minimum time simulation	2.526	2.403	1.000	1.372
Standard deviation	4.412	4.010	1.000	1.415
Convergence failure (%)	1.2%	1.2%	0.0%	0.0%

Cooley and Hansen (1989) model

	Maliar and Maliar: *standard* implementation k censored	Maliar and Maliar: *adaptable* moving bounds k censored		Maliar and Maliar: *adaptable* moving bounds k and N censored	Estimated Log-linear approach	Standard Homotopy approach
	$\lambda = 0.3$ $a = 0.0035$	$\lambda = 0.3$ $a = 0.0035$	$\lambda = 0.01$ $a = 0.0003$	$\lambda = 0.3$ $a = 0.0035$	$\lambda = 0.3$	$\lambda = 0.3$
Average computational time	4.382	3.416	3.923	3.069	1.000	2.296
Maximum time simulation	4.406	3.918	3.139	3.161	1.000	1.881
Minimum time simulation	5.046	3.907	3.303	3.167	1.000	2.854
Standard deviation	6.909	3.822	3.800	2.748	1.000	1.337
Convergence failure (%)	95.6%	84%	78.8%	80.8%	0.0%	3.6%

Notes: Summary statistics for 250 simulations, each of size T=5000. Figures are expressed in relative terms (ratio to the method scoring minimum time) unless otherwise stated.